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Primary Decomposition of Divisorial Ideals in Mori Domains

EVAN G. HOUSTON AND THOMAS G. LUCAS

Department of Mathematics, University of North Carolina at Charlotte, Charlotte, North Carolina 28223

AND

T. M. VISWANATHAN*

Universidade Estadual de Campinas, Instituto de Matemática e Estatistica, Caixa Postal 6165, 13081 Campinas, Sao Paulo, Brazil

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DEDICATED TO THE MEMORY OF SRINIVASA RAMANUJAN ON THE OCCASION OF THE 100TH ANNIVERSARY OF HIS BIRTH

1. INTRODUCTION

Mori domains have received a good deal of attention in the literature recently [1–5, 10, 13]. A Mori domain is an integral domain which satisfies the ascending chain condition on divisorial ideals. The best-known examples are (arbitrary) Noetherian domains and Krull domains, but other examples do exist [1, 2, 4].

As one would expect from the definition, many results on Noetherian domains have analogues in the Mori case. The goal of this paper is to study primary decomposition of divisorial ideals in Mori domains. In the next section we prove that a divisorial ideal in a Mori domain has only finitely many associated primes. We also introduce and study the notion of a d-irreducible ideal in a Mori domain R: the ideal I of R is d-irreducible if it is divisorial and cannot be written as the intersection of two properly larger divisorial ideals of R. We show that every d-irreducible ideal is of the form $(a):b$ and has a unique maximal associated prime. We also give examples of d-irreducible ideals.

* Currently visiting the University of North Carolina at Charlotte.

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In Section 3 we record some simple observations regarding primary decomposition in a Mori domain, and we characterize those Mori domains in which every divisorial ideal is a finite intersection of *divisorial* primary ideals. These are precisely the Mori domains in which every d-irreducible ideal is primary, or equivalently, every divisorial prime ideal has height one.

Section 4 is concerned with the notion of descent of primary decomposition. That is, we study the situation where existence of a primary decomposition of an ideal which properly contains the ideal I guarantees the existence of a primary decomposition of I. We first study descent in the general case and then apply it to the case of Mori domains. We prove that every divisorial ideal of a two-dimensional Mori domain has a primary decomposition.

In the final section, we present several examples, all of which serve to illustrate differences between Noetherian domains and general Mori domains. For instance, in Example 5.4 we produce a Mori domain with an infinite descending chain of divisorial prime ideals. Our final example shows that divisorial ideals in Mori domains do not in general have primary decompositions. We produce this example by first proving that Mori domains in which every divisorial ideal does have a primary decomposition must have a certain Krull intersection property. Via a $D+M$-type construction, we essentially reduce the question to the case of Krull domains, and finally, using an example suggested to us by Paul Eakin, we show that Krull domains do not necessarily have the required property.

If R is a domain and I is a fractional ideal of R, we denote by I_v the fractional ideal $(I^{-1})^{-1} = R:(R:I)$. This defines the so-called v-operation, and we shall freely use its properties [6, Sect. 32]. Recall that I is divisorial $\Leftrightarrow I = I_v$. Below, we list for easy reference several facts which we shall need in the sequel.

PROPOSITION 1.1. *Let R be a domain, let I be a fractional ideal of R, and let S be a multiplicatively closed subset of R.*

(i) *If I is divisorial, then so is $I:J$ for any fractional ideal J of R.*

(ii) *If I is finitely generated, then $I^{-1}R_S = R_S:IR_S$ and $R_S:(R_S:I_vR_S) = R_S:(R_S:IR_S)$.*

(iii) *R is a Mori domain $\Leftrightarrow$ for each fractional ideal A of R there is a finitely generated fractional ideal $B \subseteq A$ with $A^{-1} = B^{-1}$.*

(iv) *If R is a Mori domain and I is divisorial, then IR_S is divisorial in R_S.*

(v) *Let P be a divisorial prime ideal of R, and suppose that R has* a.c.c. *on divisorial ideals contained in P. Then R_P is a Mori domain, and divisorial ideals of R_P contract to divisorial ideals of R.*

(vi) *If R is a locally finite intersection of Mori domains, then R is a Mori domain.*

(vii) *If R is a Mori domain and $I_1 \supseteq I_2 \supseteq \cdots$ is a decreasing sequence of divisorial ideals of R with nonzero intersection, then the sequence stabilizes.*

Proof. (i) See [6, Exercise 1, p. 406].

(ii) This is [14, Lemma 4].

(iii) [11, I, Théorème 1].

(iv) Applying (iii) to I^{-1}, we have $I = I_v = A^{-1}$ with A finitely generated. Thus by (ii) $IR_S = A^{-1}R_S = R_S : AR_S$, which is divisorial.

(v) Let J' be divisorial in R_P, $J = J' \cap R$. Since R has a.c.c. on divisorial ideals contained in P, we may write $J = A_v$ with A finitely generated, $A \subseteq J$. Then $J_v \subseteq J_v R_P \cap R = A_v R_P \cap R \subseteq R_P : (R_P : AR_P) \cap R \subseteq J' \cap R = J$. Thus divisorial ideals of R_P contract to divisorial ideals of R. It follows easily that R_P is a Mori domain.

(vi) [11, I, Théorème 2].

(vii) [11, I, Théorème 1].

2. Associated Primes

For an ideal I of a domain R, we denote by $\operatorname{Ass}(I)$ the set of prime ideals of the form $I:x$, $x \in R - I$. In [2, Proposition 2.2] it is shown that, if I is a divisorial ideal in a Mori domain R, then I is contained in only finitely many maximal divisorial ideals. Our first result is a generalization of this.

THEOREM 2.1. *If I is a divisorial ideal in a Mori domain R, then the set of divisorial prime ideals containing I is finite. In particular,* $\operatorname{Ass}(I)$ *is finite.*

Proof. Let $\mathscr{F} = \{P\colon P \text{ is a divisorial prime of } R \text{ and } I \subseteq P\}$, and suppose that $\mathscr{F}$ is infinite. Since R is a Mori domain, we may choose a sequence $P_1, P_2, \ldots$ of elements of $\mathscr{F}$ such that P_1 is maximal in $\mathscr{F}$ and P_{n+1} is maximal in $\mathscr{F} - \{P_1, \ldots, P_n\}$ for $n \geqslant 1$. Consider the following chain of divisorial ideals: $P_1 \supseteq P_1 \cap P_2 \supseteq P_1 \cap P_2 \cap P_3 \supseteq \cdots$. By Proposition 1.1(vii) this chain stabilizes, so we have $P_1 \cap \cdots \cap P_n = P_1 \cap \cdots \cap P_n \cap P_{n+1}$ for some n. It follows that, for some i, $1 \leqslant i \leqslant n$, $P_i \subseteq P_{n+1}$, a contradiction. Hence $\mathscr{F}$ is finite. The "in particular" statement follows from Proposition 1.1(i).

Let I denote an ideal of a ring R. Following Kaplansky [9] we denote by $Z(I)$ the zero divisors on R/I, that is, the set of elements $x \in R$ such that $xy \in I$ for some $y \in R - I$. Ideals maximal within $Z(I)$ are prime, the so-called maximal primes of I. In the Noetherian case, these are finite in number and are elements of $\operatorname{Ass}(I)$ [9, Theorem 80]. We record a similar result. Denote by $\operatorname{Maxass}(I)$ the maximal elements of $\operatorname{Ass}(I)$.

COROLLARY 2.2. *If I is a divisorial ideal in a Mori domain R, then the set of maximal primes of I is equal to* $\operatorname{Maxass}(I)$.

Proof. If $x \in Z(I)$ then $x \in I{:}a$ for some $a \notin I$. Since $I{:}a$ is divisorial, we may expand it to an ideal P maximal in $\{I{:}b{:}b \notin I\}$. By [9, Theorem 6] P is prime, whence $P \in \operatorname{Ass}(I)$. Hence $Z(I) \subseteq \bigcup\{P\colon P \in \operatorname{Ass}(I)\}$. Since $\operatorname{Ass}(I)$ is finite, the result now follows easily from the prime avoidance lemma [9, Theorem 81].

In our next result, we record the fact that the similarity to the Noetherian case extends to minimal primes. The first conclusion in the following proposition was also observed by Roitman [13].

PROPOSITION 2.3 (cf. [9, Theorems 86 and 88]). *If I is a divisorial ideal in a Mori domain R, then I has only finitely many minimal primes, and each is an element of* $\operatorname{Ass}(I)$.

Proof. Let P be minimal over I. Then PR_P is the unique prime ideal of R_P containing the divisorial ideal IR_P. It follows that $PR_P \in \operatorname{Maxass}(IR_P)$, $PR_P = IR_P{:}\,x$, $x \in R_P - IR_P$. In particular, PR_P is divisorial in R_P. By Proposition 1.1 there is a finitely generated ideal J of R with $J \subseteq P$ and $P_v = J_v$. Hence $P_v R_P = J_v R_P \subseteq R_P{:}(R_P{:}JR_P) \subseteq PR_P$, again by Proposition 1.1. Hence $P = P_v = J_v$. Also, since $xPR_P \subseteq IR_P$, there is an $s \in R - P$ with $sx \in R$ and $sxJ \subseteq I$, whence $sxP = sxJ_v \subseteq I$. It follows that $P = I{:}sx$, $sx \notin I$, so $P \in \operatorname{Ass}(I)$. That I has only finitely many minimal primes now follows from Theorem 2.1.

DEFINITION. A divisorial ideal I ($\neq R$) in a domain R is said to be *d-irreducible* if I is not the intersection of two properly larger divisorial ideals.

If R is a Mori domain, then by Proposition 1.1(vii) the set of divisorial ideals which properly contain a divisorial ideal I has minimal elements. Clearly, I is d-irreducible $\Leftrightarrow$ this set has a minimum element. This minimum element is called the *cover* of I.

PROPOSITION 2.4. *Let I be a divisorial ideal in a Mori domain R. Then I is a finite intersection of d-irreducible ideals. Moreover, each d-irreducible ideal has the form $(a){:}b$ for suitably chosen $a, b \in R$.*

Proof. The first assertion follows easily from the ascending chain condition on divisorial ideals. For the second assertion, suppose that I is d-irreducible, and let J denote its cover. Since I is divisorial, it is an intersection of principal fractional ideals Ru (u in the quotient field of R). For at least one such u, we must have $J \nsubseteq Ru \cap R$. Hence $I = Ru \cap R = (a):b$ if $u = a/b$.

As a corollary we record a slight generalization [12, II, Théorème 1].

COROLLARY 2.5. *If P is a divisorial prime ideal in a Mori domain R, then for each nonzero element $a \in P$, there is an element $b \in R - (a)$ with $P = (a):b$.*

Proof. By Proposition 2.4 we may write $P = (c):d$, $d \in R - (c)$. It follows that $P = (a):ad/c$, ad/c being in R since $a \in P$.

PROPOSITION 2.6. *Every divisorial ideal of a Krull domain has the form $(a):b$.*

Proof. Let I be a divisorial ideal in the Krull domain R. By the approximation theorem for Krull domains, [6, Theorem 44.1], there is an element a of R which generates I at each minimal prime of I. We may then choose $b \in R$ such that b is a unit in R_P for each minimal prime P of I and such that $bR_Q = aR_Q$ for each minimal prime Q of a which does not contain I. It follows that $I = (a):b$.

PROPOSITION 2.7. *If I is a divisorial ideal in a Mori domain R, then $I = \bigcap \{IR_P \cap R\colon P \in \mathrm{Maxass}(I)\}$. If, in addition, I is d-irreducible, then $\mathrm{Maxass}(I)$ consists of a single prime M, and $I = IR_M \cap R$.*

Proof. Let $x \in R - I$. Then $I:x \subseteq Z(I)$, whence $I:x \subseteq P$ for some $P \in \mathrm{Maxass}(I)$, so that $x \notin IR_P$. It follows that $I \supseteq \bigcap \{IR_P \cap R\colon P \in \mathrm{Maxass}(I)\}$. The other inclusion is trivial. Now assume that I is d-irreducible. It suffices to show that $\mathrm{Maxass}(I)$ has only one element. I is the intersection of the ideals $IR_P \cap R$, each of which is divisorial (by Proposition 1.1). Since I is d-irreducible, we must have $I = IR_M \cap R$ for some $M \in \mathrm{Maxass}(I)$. If $P = I:y$, $y \notin I$, is also an element of $\mathrm{Maxass}(I)$, then $Py \subseteq I = IR_M \cap R$, which, since $y \notin I$, implies that $P \subseteq M$. Hence $P = M$, and the proof is complete.

Before stating our next result, we recall a definition. An ideal I of a domain R is *v-invertible* if $(IJ)_v = R$ for some fractional ideal J of R.

PROPOSITION 2.8. *Let I be a d-irreducible ideal in a Mori domain R. If J is the cover of I and M is the unique maximal prime of I, then*

(i) $J \subseteq I:M$,

(ii) $\mathrm{Ass}(I) = \mathrm{Ass}(I:M) \cup \{M\} \subseteq \mathrm{Ass}(J) \cup \{M\}$, *and*

(iii) *if I is v-invertible, then* $J = I:M = (IM^{-1})_v$.

Proof. (i) Write $M = I:x$, $x \in R - I$. Then $x \in (I:M) - I$, so $I \subsetneq I:M$. Since $I:M$ is divisorial, $J \subseteq I:M$.

(ii) If $I = M$ there is nothing to prove. Hence we assume $I \subsetneq M$. Let $P = I:y \in \mathrm{Ass}(I)$. We may assume $P \subsetneq M$. Then $y \notin I:M$. It follows that $P = I:My = (I:M):y = J:y$, so $P \in \mathrm{Ass}(I:M) \cap \mathrm{Ass}(J)$. Conversely, suppose that $Q \in \mathrm{Ass}(I:M)$. Then $Q = I:Mu$, $u \in R - (I:M)$. Hence $QMu \subseteq I = IR_M \cap R$, and $Mu \nsubseteq I$, so $Q \subseteq M$. We may assume $Q \subsetneq M$. Choose $a \in M - Q$. Then $Q = I:Mu \subseteq I:au \subseteq (I:M):au = (I:Mu):a = Q:a = Q$, whence $Q = I:au \in \mathrm{Ass}(I)$.

(iii) Clearly $I \subsetneq IM^{-1} \subseteq I:M$, so that $J \subseteq (IM^{-1})_v \subseteq I:M$. Thus we need only show that $I:M \subseteq J$. Now $IJ^{-1}J \subseteq I = IR_M \cap R$. Since $J \subseteq I$ and $IJ^{-1} \subseteq II^{-1} \subseteq R$, we have $IJ^{-1} \subseteq M$. Hence $IJ^{-1}(I:M) \subseteq M(I:M) \subseteq I$. Thus $J^{-1}(I:M) = (I^{-1}I)_v J^{-1}(I:M) \subseteq (I^{-1}IJ^{-1}(I:M))_v \subseteq (I^{-1}I)_v = R$, whence $I:M \subseteq J_v = J$.

This completes the proof.

Which ideals in a Mori domain are d-irreducible? Of course, divisorial primes are d-irreducible. The following proposition gives another type of d-irreducible ideal. We shall give yet another example in Section 5.

PROPOSITION 2.9. *Let M be a divisorial prime ideal in a Mori domain R. Then, for each nonzero element $a \in M$, $aR_M \cap R$ is d-irreducible with unique maximal prime M.*

Proof. Let $I = aR_M \cap R$. Then I is divisorial by Proposition 1.1. By Corollary 2.5 $M = (a):b$ for some $b \in R - (a)$. It follows that $M = I:b$, $b \notin I$, so $M \in \mathrm{Ass}(I)$. To show that I is d-irreducible, it suffices to show that $b \in J$, for every divisorial ideal J properly containing I. As in the proof of the preceding proposition, $IJ^{-1} \subseteq M$. Thus $aJ^{-1} \subseteq M$, so $aJ^{-1}b \subseteq (a)$, whence $J^{-1}b \subseteq R$ and $b \in J_v = J$. Hence I is d-irreducible, and since $I = IR_M \cap R$, M is the unique maximal prime of I.

From Propositions 2.8(iii) and 2.9, we easily get the following result.

COROLLARY 2.10. *If (R, M) is a quasi-local Mori domain with M divisorial, then, for each nonzero element b of R, (b) is d-irreducible with cover* $bM^{-1} = (b):M$.

3. Primary Decomposition

In this section we record some elementary facts about primary decomposition of divisorial ideals in Mori domains. As usual, a primary decomposition is a finite irredundant intersection of primary ideals having distinct radicals.

Proposition 3.1. *Let $I = Q_1 \cap \cdots Q_n$ be a primary decomposition of the divisorial ideal I in a Mori domain R. Then* $\mathrm{Ass}(I) = \{\mathrm{rad}\, Q_i : i = 1, ..., n\}$.

Proof. Standard arguments [15, Theorem 6, p. 210] show that $\mathrm{Ass}(I) \subseteq \{\mathrm{rad}\, Q_i\}$ and that, if $P = \mathrm{rad}\, Q_i$, then $P = \mathrm{rad}(I{:}x)$ for some $x \in R - I$. It remains to show that $P \in \mathrm{Ass}(I)$. However, since P is minimal over the divisorial ideal $I{:}x$, there is by Proposition 2.3 an element $y \in R - (I{:}x)$ such that $P = (I{:}x){:}\, y$. Hence $P = I{:}\ xy \in \mathrm{Ass}(I)$.

The minimal primes present no problem in a primary decomposition. We record this precisely in the following proposition, whose routine verification we omit.

Proposition 3.2. *Let I be a divisorial ideal in a Mori domain R, and let P be a minimal prime of I. Then $IR_P \cap R$ is a P-primary divisorial ideal and is the P-primary component in every primary decomposition of I.*

Our next result explains why one does not require the primary components to be divisorial.

Proposition 3.3. *The following statements are equivalent in a Mori domain R:*

(i) *Every divisorial prime ideal has height* 1.

(ii) *Every d-irreducible ideal is primary.*

(iii) *Every divisorial ideal is a finite intersection of divisorial primary ideals.*

(iv) *Principal ideals have no embedded primes.*

Proof. (i) ⇒ (ii). Let I be d-irreducible with unique maximal prime M. By (i) M is minimal over I, so $M = \mathrm{rad}(I)$. Since $I = IR_M \cap R$, I is M-primary.

(ii) ⇒ (iii). This follows from Proposition 2.4.

(iii) ⇒ (iv). Let P, $M \in \mathrm{Ass}(Ra)$, $P \subseteq M$. We shall show that $P = M$. Let $I = aR_M \cap R$. By (iii) I is a finite intersection of divisorial primary ideals. However, by Proposition 2.9, I is d-irreducible with unique maximal prime M. Therefore, I is M-primary. Since $I \subseteq P$, we must have $P = M$.

(iv) $\Rightarrow$ (i). Suppose that M is a divisorial prime of R with $htM > 1$. Then M properly contains a prime $Q \neq 0$. Choose a nonzero element a of Q and shrink Q to a prime P minimal over a. Then P is divisorial, $P \neq M$ and P, $M \in \operatorname{Ass}(Ra)$ by Corollary 2.5.

This completes the proof.

4. Descent of Primary Decomposition

The main idea in this section was motivated by the following attempt to prove that every proper divisorial ideal of a Mori domain has a primary decomposition. If not, let R be a "bad" Mori domain, and let I be a maximal (divisorial) offender. Then I is d-irreducible with unique maximal prime, say, M. Since $I \subsetneq I:M$, $I:M$ has a primary decomposition. By Zorn's lemma the set $\{A : A$ is an ideal of R such that $I = (I:M) \cap A\}$ has maximal elements. If one of these elements happens to be primary, we have a contradiction.

Of course, this approach fails in the general case, as is shown in the next section. Nevertheless, this notion of descent of primary decomposition does lead to a proof that divisorial ideals in two-dimensional Mori domains have primary decompositions (Theorem 4.6). Moreover, descent may be of some independent interest (see Proposition 4.4). We therefore digress in order to study the general case.

Definition. Let $I \subseteq J$ be ideals in a ring R. Then primary decomposition *descends from J to I* if J has a primary decomposition and $I = J \cap Q$ for some primary ideal Q of R.

Lemma 4.1. *Let I be an ideal of the ring R, let M be a prime ideal of R, and assume that $M \in \operatorname{Maxass}(I)$, $I \subsetneq M$. Suppose that $I = Q_1 \cap \cdots \cap Q_n$ is a primary decomposition of I, with Q_1 M-primary. Then $I:M$ has a primary decomposition which includes the primary components $Q_2, \ldots, Q_n$. (Thus primary decomposition descends from $I:M$ to I.)*

Proof. Clearly, $I:M = (Q_1 : M) \cap (Q_2 : M) \cap \cdots \cap (Q_n : M)$. For $i > 1$, Q_i is P-primary for some $P \in \operatorname{Ass}(I)$, $P \neq M$. Since $M \in \operatorname{Maxass}(I)$, $Q_i : M = Q_i$ for $i > 1$. If $Q_1 = M$ then $I:M = Q_2 \cap \cdots \cap Q_n$. Otherwise, $Q_1 : M$ is M-primary, and the only possible redundancy is that $Q_1 : M \supseteq Q_2 \cap \cdots \cap Q_n$. This completes the proof.

It is convenient to introduce the following notation. Let R be a ring, I an ideal of R, and M a prime ideal of R with $M \in \operatorname{Ass}(I)$. Denote by $\mathscr{Q}(I, M)$

the set $\{Q : Q$ is an ideal of R and Q is maximal among those ideals A of R for which $I = (I:M) \cap A\}$.

Remark. Note that $Q \subseteq M$ if $Q \in \mathcal{Q}(I, M)$. This follows because, if $M = I:x$, then $xQ \subseteq (I:M) \cap Q = I$, so $Q \subseteq I:x = M$.

LEMMA 4.2. *Let I be an ideal of a ring R, and let M be a prime ideal of R with $M \in \operatorname{Ass}(I)$. If $Q \in \mathcal{Q}(I, M)$ then either $M = \operatorname{rad} Q$ or there is an element $a \in R$ and an infinite ascending chain $Q: a \subsetneq Q: a^2 \subsetneq \cdots$.*

Proof. By the remark above, $Q \subseteq M$. If $M \neq \operatorname{rad} Q$ choose $a \in M$ with $a^n \notin Q$ for $n = 1, 2, \ldots$. By definition $I \subsetneq I\colon M \cap (Q, a^n)$. Choose $b_n = q_n + r_n a^n$ with $b_n \in (I:M) - I$, $q_n \in Q$, $r_n \in R$. We claim that $r_n \in (Q:a^{n+1}) - (Q:a^n)$. Clearly, $r_n \notin Q:a^n$. However, $b_n a = q_n a + r_n a^{n+1}$. Since $b_n a \in I \subseteq Q$, $r_n a^{n+1} \in Q$, as claimed.

Remark. The sequence $\{Q:a^n\}$ is a crucial consideration in the key result involved in proving that Noetherian rings have primary decomposition [15, Lemma 2, p. 209]. In fact, our notion of descent can be used to give an alternate proof of this result. We pause to record this observation.

PROPOSITION 4.3 [15, Lemma 2, p. 209]. *In a Noetherian ring, every irreducible ideal is primary.*

Proof. If the result is false, let I be maximal among irreducible non-primary ideals. The set $\{I\colon a : a \notin I\}$ has maximal elements, and these maximal elements are prime. Let $M = I:x$ be one of these. Of course, $I \subsetneq I:M$. Pick $Q \in \mathcal{Q}(I, M)$. By irreducibility of I, $Q = I$. By Lemma 4.2 $M = \operatorname{rad} I$. Hence M is the only associated prime of I, from which it follows that I is M-primary.

We now combine Lemmas 4.1 and 4.2 to yield our main result on descent.

PROPOSITION 4.4. *Let I be an ideal of the ring R, and let M be a maximal ideal of R, with $M \in \operatorname{Ass}(I)$, $I \subsetneq M$. Then I has a primary decomposition $\Leftrightarrow$ $I:M$ has a primary decomposition and $\exists Q \in \mathcal{Q}(I, M)$ such that $(Q:a) \subseteq (Q:a^2) \subseteq \cdots$ is stationary for every $a \in R$.*

Proof. If I has a primary decomposition, then by Lemma 4.1 $I:M$ also has a primary decomposition and $I = (I:M) \cap Q'$ for some primary ideal Q'. Then, in fact, Q' is M-primary. To see this, pick $x \in (I:M) - Q'$. Then $Mx \subseteq I \subseteq Q'$, $x \notin Q'$, implies that $M \subseteq \operatorname{rad} Q'$. Hence $M = \operatorname{rad} Q'$ and Q' is M-primary. Expand (by Zorn's lemma) Q' to an element $Q \in \mathcal{Q}(I, M)$. Then rad $Q = M$ (by the remark immediately preceding Lemma 4.2), and this part of the result follows.

Conversely, if Q is as described, then, since $I = (I:M) \cap Q$, we need only show that Q is primary. However, by Lema 4.2, $M = \text{rad}\, Q$, so Q is M-primary since M is maximal.

PROPOSITION 4.5. *Let I be an ideal of the ring R, and let M be a maximal ideal of R with $M \in \text{Ass}(I)$. Assume that $M = \text{rad}(I, a)$ for some $a \in M$ and that the chain $I:a \subseteq I:a^2 \subseteq \cdots$ stabilizes. If $I:M$ has a primary decomposition, then primary decomposition descends from $I:M$ to I.*

Proof. By assumption $I:a^k = I:a^{k+1}$ for some k. Since (I, a^k) is M-primary, it suffices to show that $I = (I:M) \cap (I, a^k)$. Suppose $x = b + ra^k$ with $x \in I:M$, $b \in I$, $r \in R$. Then $xa = ba + ra^{k+1}$, and $ra^{k+1} = xa - ba \in I$, whence $r \in I:a^{k+1} = I:a^k$. Therefore, $x \in I$, as desired.

We close this section by proving the promised result on Mori domains. The main tool is Proposition 4.5.

THEOREM 4.6. *Let R be a Mori domain in which each divisorial prime ideal M satisfies one of the following conditions*:

(i) *MR_M is the radical of a principal ideal in R_M.*

(ii) *Height $M \leqslant 2$.*

(iii) *R_M is Noetherian.*

Then every divisorial ideal of R has a primary decomposition.

Proof. Deny the conclusion and let I be maximal among those divisorial ideals which do not have primary decompositions. Then I is d-irreducible. Let M be its unique maximal prime. Of course, $I \subsetneq I:M$, and, since $I = IR_M \cap R$, $I:M = (I:M)R_M \cap R$ also. By choice of I, $I:M$ has a primary decomposition, so it suffices to show that primary decomposition descends from $I:M$ to I.

From this discussion it is clear that if suffices to prove the result locally. That is, we may assume that (R, M) is a quasi-local Mori domain, that M satisfies one of the three conditions listed, that I is a divisorial ideal of R with $M \in \text{Ass}(I)$, and that $I:M$ has a primary decomposition. Now if M satisfies condition (iii), there is nothing to prove, and, if M satisfies condition (i), the result follows from Proposition 4.5. Suppose that M satisfies condition (ii): Since I is contained in only finitely many divisorial primes, M contains an element a which lies in no other divisorial prime containing I. It follows that $M = \text{rad}(I, a)$, and the result again follows from Proposition 4.5.

5. Examples

We begin this section with another class of examples of d-irreducible ideals.

PROPOSITION 5.1. *Let (R, M) be a quasi-local Mori domain, with M finitely generated. Assume that $M^{-1} = R[x]$ for some $x \in M^{-1} - R$. Then $I = xM$ is d-irreducible with unique maximal prime M.*

Proof. Since M^{-1} is a ring, M is not principal. Hence $xM \subseteq M$; we assume that the containment is proper, for otherwise there is nothing to prove. Thus $1/x \notin M^{-1}$. Since M is finitely generated and $xM \subseteq M$, x is integral over R. Consider an integrality equation over R of minimal degree, say $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$. If we put $t = x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$, then $tx \in R$ but $t \notin R$. We shall show that $tx \in J$ for every divisorial ideal J of R which properly contains I.

To this end let $u \in J^{-1}$; it suffices to show that $utx \in R$. Since $uxM \subseteq R$, $ux \in M^{-1} = R[x]$. Write $ux = r + xz$, with $r \in R$, $z \in R[x]$. Then $utx = rt + ztx$. Since $t \cdot 1/tx = 1/x \notin R[x]$, we have $tx \in M$. Hence $txz \in M$, and it now suffices to show that $r \in M$. Choose $a \in J - xM$. Then $uxa = ra + xza \in M$. Since $ua \in R$ *and* $uxa \in M$, we have $ua \in M$, whence $ra = x(ua - za) \in xM$. As $a \notin xM$, $r \in M$, as desired. Therefore, I is d-irreducible. Finally, since $M = xM : tx$, $M \in \mathrm{Maxass}(I)$.

A simple example of the situation described in Proposition 5.1 is the ring $R = k[[X^2, X^3]]$ with maximal ideal $M = (X^2, X^3)$.

The following result is in sharp contrast to Proposition 5.1.

PROPOSITION 5.2. *Let (R, M) be a quasi-local Mori domain, with M finitely generated but not principal. If $b \in M$ then bM is not d-irreducible.*

Proof. By Corollary 2.5 $M = (b):c$ for some $c \in R - (b)$. Thus $bM = bR \cap R(b^2/c) = bR \cap (R(b^2/c) \cap R)$. If bM is d-irreducible, we must have $bM = R(b^2/c) \cap R = (b^2):c$. We shall show that this is impossible. Set $x = c/b$ and note that $xM \subseteq M$ (since M is not principal). Hence x is integral over R. As above consider an integrality equation over R of minimal degree: $x^n + r_{n-1}x^{n-1} + \cdots + r_0 = 0$, $n > 1$. We claim that $c^i \in (b^{i-1})$ for $i \geqslant 1$. Since $c \in M$, $c^2 \in (b)$. Inductively, assume that $c^i \in (b^{i-1})$ for $i = 1, 2, \ldots, m$. Write $c^m = ab^{m-1}$. If $a \notin M$ then $b^{m-1} \in (c^m) = c(c^{m-1}) \subseteq c(b^{m-2})$, whence $b \in (c)$. However, this contradicts the fact that M is not principal. Hence $a \in M = (b):c$. Thus $c^{m+1} = acb^{m-1} \in (b^m)$, as claimed. It follows that $b^2x^i \in cR$ for $i \geqslant 1$. Multiply the integrality equation above by b^2. This yields $b^2x^n + r_{n-1}b^2x^{n-1} + \cdots + r_1b^2x \in (b^2)$. Since $(b^2/c)x^i \in R$ for

$i \geq 1$, we have $(b^2/c)x^n + r_{n-1}(b^2/c)x^{n-1} + \cdots + r_1(b^2/c)x \in (b^2):c = bM$. Hence $(b/c)x^n + r_{n-1}(b/c)x^{n-1} + \cdots + r_1(b/c)x = m \in M \subseteq R$. However, we then have $x^{n-1} + r_{n-1}x^{n-2} + \cdots + r_1 - m = 0$, contradicting the minimality of n. Hence bM is not d-irreducible.

In spite of Proposition 5.2, ideals of the form bM can be d-irreducible, as the following example shows. The example is essentially that of [9, Exercise 8, p. 114] and was also studied by Barucci and Gabelli [2].

EXAMPLE 5.3. Let k be a field, and let $\{Y\} \cup X$ be algebraically independent over k, where $X = \{X_i\}$ is a nonempty set of indeterminates. Let $R = k + Yk[X, Y]$. Then

(i) R is an integrally closed Mori domain,

(ii) $M = Yk[X, Y]$ is a maximal divisorial (maximal) ideal of R, and

(iii) YM is d-irreducible.

Proof. (i) By [1, Example 3.8(b)], $T = k + Yk[X, Y]_{Yk[X,Y]}$ is an integrally closed Mori domain, and it is easy to see that $R = k[X, Y] \cap T$. Thus R is an integrally closed Mori domain by Proposition 1.1.

(ii) Clearly, $M = R:X_i$ for any $X_i \in X$. Thus M is divisorial. It is also clear that M is maximal.

(iii) Let J be a divisorial ideal of R which properly contains YM. It suffices to show that $Y \in J$. We first show that $J \subseteq M$. Let $R' = k[X, Y]$. If JR' is contained in a height one prime P of R', then, since $YM \subseteq J$, we must have $P = YR'$. Thus $J \subseteq YR' \cap R \subseteq M$. If JR' is not contained in a height one prime of R', then $R':JR' = R'$ [6, Corollary 44.8]. Thus $J^{-1} \subseteq R'$: $JR' = R' = M^{-1}$, whence $J = J_v \supseteq M_v = M$. Since M is maximal, we have $J \subseteq M$ in this case as well. Now let $u \in J^{-1}$; we shall show that $uY \in R$. Since $uYM \subseteq R$, $uY \in M^{-1}$. Let $a \in J - YM$. Since $J \subseteq M$, we may write $a = Yt$ for some $t \in M^{-1} - R$. Hence $uYt = ua \in R$. However, it is easily shown that the product of two elements of $M^{-1} - R$ again lies in $M^{-1} - R$. Since $t \notin R$, this gives $uY \in R$, as desired.

If X is infinite in the example above, then M is a divisorial prime of infinite height. By modifying the example, we can in fact produce an infinite decreasing sequence of *divisorial* prime ideals, which is perhaps surprising in light of the Noetherian situation.

EXAMPLE 5.4. Let K be a field and $X_0, X_1, X_2, \ldots$ be indeterminates. Define a descending sequence of rings as follows:

$$D_0 = K[X_0, X_1, X_2, \ldots],$$
$$D_1 = K[X_0, \{X_0 X_1^{i_1}\}, X_2, \ldots],\ i_1 \geqslant 1,$$
$$D_2 = K[X_0, \{X_0 X_1^{i_1}\}, \{X_0 X_1^{i_1} X_2^{i_2}\}, X_3, \ldots],\ i_1, i_2 \geqslant 1,$$
$$\vdots$$
$$D_n = K[X_0, \{X_0 X_1^{i_1}\}, \ldots, \{X_0 X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n}\}, X_{n+1}, \ldots],$$
$$i_1, i_2, \ldots, i_n \geqslant 1,$$
$$\vdots$$
$$R = K[X_0, \{X_0 X_1^{i_1}\}, \ldots, \{X_0 X_1^{i_1} \cdots X_n^{i_n}\}, \ldots],\ i_j \geqslant 1 \text{ for all } j.$$

Then we have

(i) $R = \bigcap_{n=0}^{\infty} D_n$.

(ii) Each D_n is a Mori domain.

(iii) For each $n \geqslant 1$, $P_n = R \cap X_n D_0$ is a divisorial prime ideal of R. Moreover, $M = R \cap X_0 D$ is maximal and divisorial and $M \supset P_1 \supset P_2 \supset \cdots$ is an infinite descending chain of divisorial prime ideals.

(iv) The divisorial ideals of R contained in M satisfy the ascending chain condition.

(v) R_M is a quasi-local Mori domain with an infinite descending chain of divisorial prime ideals.

Proof. (i) Clearly, $R = \bigcap_{n=0}^{\infty} D_n$.

For the proof of (ii), define for each $r \geqslant 1$, $E_r = K[X_0, X_1, \ldots, X_{r-1}, \{X_{r-1} X_r^{i_r}\}, X_{r+1}, \ldots]$, $i_r \geqslant 1$. Then for $n \geqslant 1$, it is easily seen that $D_n = \bigcap_{r=1}^{n} E_r$. As each E_r is a Mori domain (by Example 5.3 and [10, Théorème 5]) and a finite intersection of Mori domains is Mori (Proposition 1.1), we have that each D_n is a Mori domain.

(iii) It is clear that each P_n is a prime ideal of R and that M is a maximal ideal. From the definition of R, it is apparent that $M \supsetneqq P_1 \supsetneqq P_2 \supsetneqq \cdots$ is an infinite descending chain of prime ideals. To see that each of these is divisorial, observe that $M = (X_0) : X_0 X_1$ and $P_n = (X_0 X_1 \cdots X_n) : X_0 X_1 \cdots X_n X_{n+1}$ for each $n \geqslant 1$.

To prove (iv) let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of divisorial ideals of R contained in M. As M is a subset of $X_0 D_0$, each element of M is divisible by X_0 in D_0. Hence the same holds for every element of $\bigcup_{n=1}^{\infty} I_n$. However, as every element of R is a polynomial, no element is a multiple of every X_j. Thus there is a least integer $k \geqslant 0$ such that every element of $\bigcup_{n=1}^{\infty} I_n$ is a multiple of X_k in D_0 and some element, say t, is not a multiple of X_{k+1}. As $t \in \bigcup_{n=1}^{\infty} I_n$ and $I_j \subseteq I_{j+1}$ for all j, we have $t \in I_m$ for all $m \geqslant p$, for some fixed p.

To complete the proof of (iv), we first extend each I_n to a divisorial ideal in D_{k+1} by sending I_n to $(I_n D_{k+1})_v$ where the v-operation is taken with respect to D_{k+1}. As D_{k+1} is a Mori domain, we next have that the chain $(I_1 D_{k+1})_v \subseteq (I_2 D_{k+1})_v \subseteq \cdots$ stabilizes at, say, r. Finally, from the lemma below using $c=t$, we conclude that the original chain is stable for $m \geqslant \max\{r, p\}$.

Before presenting the lemma we note that (v) follows from Proposition 1.1.

In the proof of the following lemma we denote by v_i the X_i-adic valuation of the field $K(X_0, X_1, X_2, \ldots)$ and by V_i the corresponding valuation ring.

LEMMA. *Let R and D_{k+1} be as above. Let I and J be divisorial ideals of R satisfying the following conditions*:

(i) $I \subseteq J$.

(ii) $(ID_{k+1})_v = (JD_{k+1})_v$.

(iii) *There exists an element $c \in I$ such that $v_{k+1}(c)=0$. Then $I=J$.*

Proof. Assume that $I \neq J$. Then there exist $a \in J \backslash I$ and $b \in I^{-1} \backslash J^{-1}$ such that $ab \notin R$. As $(ID_{k+1})_v = (JD_{k+1})_v$ and $I^{-1} = (R:I) \subseteq (D_{k+1}:ID_{k+1})$, $ab \in D_{k+1} \backslash R$. By the definition of R and D_{k+1}, it must be that there exists $j \geqslant 2$ such that some monomial of the polynomial ab is divisible by X_{k+j} (in D_0) but not by X_{k+j-1}. Hence we have

$$v_{k+j-1}(ab)=0. \tag{1}$$

As $b \in I^{-1}$ and $c \in I$, we have that $bc = r \in R$. Moreover, by hypothesis $v_{k+1}(c)=0$ and so by definition of R, $v_{k+j-1}(c)=0$. Hence,

$$0 \leqslant v_{k+j-1}(r) = v_{k+j-1}(bc) = v_{k+j-1}(b). \tag{2}$$

Combining (1) and (2), we get $v_{k+j-1}(a) = v_{k+j-1}(b) = 0$. Thus a, b, c and r are all units in V_{k+j-1}, and so they survive when we pass to the residue class field of V_{k+j-1} where we denote the respective images as $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{r}$.

As v_i denotes the X_i-adic valuation, we may view the residue field of V_{k+j-1} as $K(X_0, X_1, \ldots, X_{k+j-2}, X_{k+j}, \ldots)$.

Let w be the degree valuation of the residue field defined by X_{k+j}. Since a, c, and r are polynomials in R, $w(\bar{a}) = w(\bar{c}) = w(\bar{r}) = 0$. Furthermore, as $r = bc$, we have $w(\bar{b}) = w(\bar{r}) - w(\bar{c}) = 0$. However, ab is divisible by X_{k+j} (and not by X_{k+j-1}) so that $w(\overline{ab}) < 0$ and we have that $0 = w(\bar{b}) = w(\overline{ab}) - w(\bar{a}) < 0$, which is impossible. Hence it must be that $I=J$.

We shall end the paper by producing a Mori domain having a divisorial ideal with no primary decomposition. In the three examples considered so far, every divisorial ideal does have a primary decomposition. This is clear in Example 5.1, and in the other two examples, the maximal divisorial ideals are radicals of principal ideals, and existence of primary decompositions follows from Theorem 4.6. Our example depends on two results, the first of which is a Krull-intersection type theorem similar to [7, Proposition 3.1].

PROPOSITION 5.5. *If R is a Mori domain in which every divisorial ideal has a primary decomposition, then the intersection of the ideals primary to any maximal divisorial ideal is zero.*

Proof. Our proof is modeled on that of [7, Proposition 3.1]. Let M be a maximal divisorial ideal of R, and let x be a nonzero element of M. Then Mx is divisorial and therefore has a primary decomposition $Mx = Q_1 \cap \cdots \cap Q_n$. Since $x \notin Mx$, $x \notin Q_i$ for some i. Since $Mx \subseteq Q_i$ this yields $M \subseteq \operatorname{rad} Q_i$, where Q_i is primary to some prime in $\operatorname{Ass}(Mx)$. However, every prime in $\operatorname{Ass}(Mx)$ is divisorial, and M is maximal divisorial. Hence $M = \operatorname{rad} Q_i$, and x is not in the M-primary ideal Q_i.

PROPOSITION 5.6. *Let T be a Krull domain containing a maximal ideal M such that the intersection of the M-primary ideals is not zero. If T contains fields $K \subsetneq L$, then $R = K + MT_M$ is a Mori domain which contains a divisorial ideal having no primary decomposition.*

Proof. R and T_M have the same maximal ideal (hence the same prime ideals), so R is a Mori domain by [1, Theorem 3.2]. $N = MT_M$ is divisorial in R, since $N = R:u$ for any $u \in L - K$. Since the N-primary ideals of R are the same as the N-primary ideals of T_M, the conclusion follows from Proposition 5.5.

There remains only the construction of a Krull domain satisfying the requirements of Proposition 5.6. The authors are indebted to Professor Paul Eakin for providing the following example.

EXAMPLE 5.7. Let $U, V, W, X_1, X_2, \ldots$ be algebraically independent over the field L. Put $R = L[U, V, W, \{X_i, X_i/U^i, (V - X_i)/W^i: i = 1, 2, \ldots\}]$, and $T = R[U^{-1}] \cap R[W^{-1}]$. Then (assuming L properly contains a field K), T is the required example. To see this note that $R[U^{-1}] = L[U, V, W, \{(V - X_i)/W^i: i = 1, 2, \ldots\}][U^{-1}]$ is a Krull domain, since $\{U, V, W\} \cup \{(V - X_i)/W^i\}$ is algebraically independent over L. Similarly, $R[W^{-1}]$ is a Krull domain. Therefore, T is a Krull domain.

We show below that the ideal $(U, W)T$ is not the unit ideal. Granting this, let M be a maximal ideal of T containing $(U, W)T$, and let Q be M-primary. Then, for some positive integer k, U^k, $W^k \in Q$, whence

$V=((V-X_k)/W^k)\cdot W^k+(X_k/U^k)\cdot U^k\in Q$. Hence V lies in every M-primary ideal of T, and T is the required example.

To show that $(U, W)T\neq T$, it suffices to show that $(U, W)S\neq S$ for some overring S of T. Let $S=L[U, W, \{V/U^jW^j: j=1, 2, \ldots\}, \{X_i/U^jW^j: i=1, 2, \ldots; j=1, 2, \ldots\}]$. Clearly, $T\subseteq S[U^{-1}]\cap S[W^{-1}]$, so it suffices to show $S[U^{-1}]\cap S[W^{-1}]\subseteq S$. Accordingly, let $f\in S[U^{-1}]\cap S[W^{-1}]$. Then f has an expression as a sum of terms of the form $cU^{n_1}W^{n_2}V^{n_3}X$, where $c\in L$, n_1, n_2, n_3 are integers with $n_3\geqslant 0$, and X is some (possibly empty) product of the X_i. Now any such term for which $n_3>0$ or for which X is a nonempty product clearly lies in S. Hence we may assume that $f=\sum c_{ij}U^{n_i}W^{n_j}$, where $(n_i, n_j)\neq(n_k, n_l)$ for $(i, j)\neq(k, l)$. We shall complete the proof by showing that each exponent is nonnegative. Since $f\in S[U^{-1}]$, U^mf is an element of the polynomial ring $L[U, W, V/W^n, \{X^i/W^n\}]$ for some fixed m, n. Thus U^mf has a canonical expression as a sum of constant multiples of products of the algebraically independent elements U, W, V/W^n, X_1/W^n, X_2/W^n, ... It is clear that in this expression the exponents of V/W^n and X_i/W^n must be zero. It then follows that in the expression above for f each exponent n_j of W must be nonnegative. A symmetric argument shows that each exponent n_i of U must be nonnegative. Therefore, $f\in S$, as desired. Finally, it is clear that $(U, W)S\neq S$.

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